constant quantity. We denote this ratio by C.

In the final stage we should consider the boundary value problem for the function $u_2(p)$: $u_2(p) = u(p) - Cu_1(p)$. The process of successive approximations for this function will lead to a convergent algorithm. The solution will be completed by transfer to the function $u(p)$.

In solving the problem of the theory of elasticity, we must begin with the six partial solutions of the boundary value problems and expand the eigenfunction obtained from the initial boundary condition in terms of the functions obtained from the partial solutions.

We note that the method described here was used in solving the second outer problem for an incompressible medium in /10/.

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SOLUTION OF THREE-DIMENSIONAL PROBLEMS OF THE THEORY OF ELASTICITY USING THE MONTE CARLO METHOD*

B.E. POBEDRYA and P.V. CHISTYAKOV

Two versions of the Monte Carlo (MC) method for solving problems of the theory of elasticity are discussed. One uses the process of random walk over spheres to solve the Lamé equations, and the other represents the quantity sought in the form of multiple integrals (e.g. when solving the Cauchy problem for the wave equation of the theory of elasticity in an unbounded space).

The process of random walk over spheres was proposed in /1/ for solving the Laplace equation, and was later used in more complicated problems (an analysis of the work done on this subject can be found in /2, 3/). A solution of the boundary value problem for the Lamé equation was studied for the plane case in /4/, and the possibility of using the MC method for the problems of flexure of plates was discussed in /3, 5/**. (** A further development of methods of solving the problems of plate flexure can be found in the paper by V.M. Ivanov and O.Yu. Kulchitskii. Development and study of effective methods of random walk over circles for solving problems of plate flexure and the plane problem of the theory of elasticity. Deposited at VINITI, No.3270-83, Leningrad, 1983.) Theorems were given in /6/, enabling the initial system of elliptic equations to be replaced by a system of integral equations which

Prikl.Matem.Mekhan., 52, 2, 341-345, 1988 can be solved conveniently using the MC method. An integral representation applicable to the Lamé equations is already known /7/.

Other papers which deal with the solution of partial differential equations, only mention the possibility of solving the Lamé equations. However, even in the plane case, only separate examples were given and the problem of substantiating the method (the empirical nature of the estimate of the solution used, the accuracy of the estimate, etc.), was given little thought. The authors are unaware of any cases where specific three-dimensional problems of the theory of elasticity have been solved using a digital computer.

1. Empirical assessment of the solution of boundary value problems for the Lamé equations. Let us consider the solution of the first boundary value problem for the Lamé equations in the region $G \subset R^3$ (Γ is the boundary of the region)

$$(1-2\nu)\,\nabla^2\mathbf{u}(P) + \operatorname{grad}\operatorname{div}\mathbf{u}(P) = 0, \quad \mathbf{u}\,|_\Gamma = \boldsymbol{\varphi}\,. \tag{1.1}$$

The mean value theory /7/ holds for the vector function $\mathbf{u}$ satisfying the Lamé equation.

The following relation holds for any sphere $S(P, R) \subset G$ of radius R with the centre at the point P:

$$\mathbf{u}(P) = \frac{1}{4\pi R^2}\int\limits_{S(P,R)} A(P,Q)\,\mathbf{u}(Q)\,dS(Q) \tag{1.2}$$

where $A(P, Q)$ is a matrix whose elements are

$$A_{ij} = A_{ij}(\varphi,\theta) = \frac{3}{8-12\nu}(5x_ix_j + (1-4\nu)\,\delta_{ij}) \tag{1.3}$$

$$x_1 = \sin\varphi\sin\theta, \quad x_2 = \cos\varphi\sin\theta, \quad x_3 = \cos\theta$$

while φ and θ are spherical coordinates of the point Q on the sphere $S(P, R)$.

We will solve problem (1.1) by the MC method, using the integral relation (1.2) and random walk over spheres.

Theorem. Let P be any point belonging to the region G, and let the random quantity $\mathbf{u}_\mu(P)$ be defined on the trajectory $P, Q_1, \ldots, Q_\mu$ of the random walk over spheres up to emergence in the ε-neighbourhood of the boundary Γ_ε by the formula

$$\mathbf{u}_\mu(P) = \begin{cases} A(P,Q_1)\ldots A(Q_{\mu-1},Q_\mu)\,\mathbf{u}(Q_\mu), & P \in G\setminus\Gamma_\varepsilon \\ \mathbf{u}(P), & P\in\Gamma_\varepsilon \end{cases} \tag{1.4}$$

where $A(Q_{k-1}, Q_k)$ is a matrix with elements $A_{ij}(\varphi_k, \theta_k)$ of the form (1.3), φ_k and θ_k are spherical coordinates of the point Q_k uniformly distributed over the sphere with centre Q_{k-1}; φ_k, θ_k $(k = 1, \ldots, \mu)$ are independent random quantities, $Q_\mu \in \Gamma_\varepsilon$ and $Q_k \notin \Gamma_\varepsilon$ for any $k < \mu$.

Then, in the case when the expectation value has a finite value,

$$\mathbf{M}(\mathbf{u}_\mu(P)) = \mathbf{u}(P)\,. \tag{1.5}$$

Proof. Let us write $\mathbf{f}(P) = \mathbf{M}(\mathbf{u}_\mu(P))$.

When $P \in \Gamma_\varepsilon$, we obviously have $\mathbf{f}(P) = \mathbf{u}(P)$.

Let now $P \in G\setminus\Gamma_\varepsilon$. According to the formula for the total expectation value, we have

$$\mathbf{f}(P) = \mathbf{M}[A(P,Q_1)\,\mathbf{M}[\mathbf{u}_\mu(Q_1)\mid Q_1]] = \mathbf{M}[A(P,Q_1)\,\mathbf{f}(Q_1)]$$

from which, using the relation

$$\mathbf{M}[A(P,Q_1)\,\mathbf{f}(Q_1)] = \frac{1}{4\pi R^2}\int\limits_{S(P,R)} A(P,Q_1)\,\mathbf{f}(Q_1)\,dS(Q_1)$$

we find that the function $\mathbf{f}(P)$ satisfies, for any $P \in G\setminus\Gamma_\varepsilon$ the integral relation (1.2).

The solution (1.5) can be written in the more convenient form

$$\mathbf{u}(P) = \sum_{k=1}^{\infty}\mathbf{P}\{\mu = k\}\,\mathbf{M}(\mathbf{u}_\mu(P)\mid\mu = k) \tag{1.6}$$

where $\mathbf{P}\{\mu = k\}$ is the probability of the appearance of the trajectory of length k, and $\mathbf{M}(\mathbf{u}_\mu(P)\mid\mu = k)$ is the conditional expectation value under the condition that the length of the trajectory is equal to k.

Let us assume that the law of distribution of the random quantity ξ depends on the parameter a. We shall call the estimate of a, any function of sampled values $\varphi(\xi_1,\ldots,\xi_N)$ used as an approximation to a. If $\mathbf{M}\varphi = a$, then the estimate φ will be called empirical. Usually, an arithmetic mean is used as the estimate of $a = \mathbf{M}\xi$. The validity follows from the law of large numbers /8/.

If $\mathbf{u}^{(1)}_{\mu(1)}, \ldots, \mathbf{u}^{(N)}_{\mu(N)}$ are independent samples of the random quantity $\mathbf{u}_\mu(P)$, then, under the conditions of the theorem, the estimate

$$\bar{\mathbf{u}}(P) = \frac{1}{N}\sum_{i=1}^{N} \mathbf{u}^{(i)}_{\mu(i)}(P) \tag{1.7}$$

will be an empirical and valid estimate of $\mathbf{u}(P)$. However, when the estimate (1.7) is used, the convergence of $\bar{\mathbf{u}}(P)$ to the solution $\mathbf{u}(P)$ as $N \to \infty$ is poor, since the growth in dispersion of the elements of the product of random matrices occurring in (1.4) exerts a substantial influence /9, 10/. The dispersion increase exponentially as t increases for the elements of the product of t independent random matrices $A(\varphi_i, \theta_i)$ $(i = 1, \ldots, t)$ of the form (1.3) with a base substantially greater than unity.

We have found that the dispersion of the elements of the product of dependent matrices A appearing in (1.4) differs little, when $\mu = t$, from (1.7) when $t > 4$. The probabilities $\mathbf{P}\{\mu = k\}$, appearing in (1.6) decrease rapidly as k increases.

2. **Two empirical assessments of the solution of the Lamé equations.** Below we give two estimates for the solution of problem (1.1). One involves rejecting "long" trajectories (*LT*), and the other involves replacing the matrix A in the *LT* by a unit matrix. The first estimate is given by the formula /9/

$$\bar{\mathbf{u}}_n(P) = \sum_{k=1}^{n} \frac{N_k}{N}\Big(\frac{1}{N_k}\sum_{i:\mu(i)=k} \mathbf{u}^{(i)}_k(P)\Big) \tag{2.1}$$

where N_k is the number of samples of length k, $\mu(i)$ is the length of the trajectory in the i-th sample. The estimate enables us to calculate the first n terms in (1.6). The terms with number $k \geqslant n$ are assumed to be equal to zero. This introduces a systematic error equal to the residue of the series (1.6) decreasing in magnitude as n increases, while the statistical error increases due to the appearance of terms corresponding to *LT*.

Using the solutions of a number of problems, we obtained two additional rules for choosing n in formula (2.1), complementing each other: 1) the number of terms in the series (1.6) included in the sum (2.1) increases as long as the RMS error is comparable in magnitude with the term; 2) n is chosen from the similar problems whose solution is known.

Use of the estimate (2.1) gives an appreciable increase in the accuracy of the solution obtained, as compared with the usual estimate (1.7) in which the *LT* are not rejected.

The second estimate of the solution $\mathbf{u}(P)$ is given by the formula

$$\bar{\mathbf{v}}_t(P) = \sum_{k=1}^{t} \frac{N_k}{N}\Big(\frac{1}{N_k}\sum_{i:\mu(i)=k} \mathbf{u}^{(i)}_k(P)\Big) + \sum_{k=t+1}^{\infty}\Big(\frac{1}{N}\sum_{i:\mu(i)=k} \mathbf{u}(Q_{\mu(i)})\Big). \tag{2.2}$$

All samples of the random quantity $\mathbf{u}_\mu(P)$ corresponding to the trajectories of length $k \leqslant t$ are found from formula (1.4), while when $k > t$, the product of the matrices A is replaced by a unit matrix. Using the numerical results for different regions, initial points, ε and ν, we found that already when $\mu = 5$, the product of the matrices A in (1.4) is fairly close to the unit matrix.

We shall consider, as an example, the solution of the boundary value problem (1.1) for a number of regions G. The value of the vector $\mathbf{u}(P)$ was found at an arbitrary point $P \in G \setminus \Gamma_\varepsilon$. The function $\boldsymbol{\varphi}(P)$ ensuring the displacement field

$$\begin{gathered}\mathbf{u}(P) = (u_x, u_y, u_z) \\ u_q = \frac{\bar{q}\bar{z}}{r^3} - (1-2\nu)\frac{\bar{q}}{r(r+\bar{z})}, \quad q = x, y \\ u_z = \frac{\bar{z}^2}{r^3} + (1-\nu)\frac{2}{r}; \quad r^2 = \bar{x}^2 + \bar{y}^2 + \bar{z}^2 \\ \bar{x} = x - a, \quad \bar{y} = y - b, \quad \bar{z} = z + c\end{gathered} \tag{2.3}$$

in G was used as the boundary conditions.

The displacement vector (2.3) represents the solution of the Boussinesq problem of an elastic half-space $z \geqslant -c$ acted upon by a concentrated force at the point $(a, b, -c)$ in the direction of the OZ axis.

We have used, as the region G_1, a sphere of radius $R = 1$ with centre at the origin of coordinates, with an excised two-sided angle perpendicular to the section XOY (Fig.1). AOB is the intersection of the two-sided angle with the plane XOY, $\operatorname{tg}\beta = 0{,}1$. We use, as the region G_2, the region consisting of a parallelepiped and a unit cube (Fig.2) $|AB| = |AD| = |DE| = 1$; $|AO| = |OC| = 2$.

The table gives the estimates $\bar{\mathbf{u}}_n$ and $\mathbf{V}_t$ of the known solutions (2.3) at the points $P_1(-0.4; 0; 0) \in G_1$ and $P_2(0.8; 0.5; 0.8) \in G_2$, calculated from formulas (2.1) and (2.2), respectively. In

(2.3) $a = b = 0, c = 1.5$ was chosen for P_1 and $a = b = c = 0.5$ for P_2. The number of samples $N = 20\,000$, $n = 9$, $t = 5$, $\nu = 0.2$. The estimates for the dispersions $\mathbf{D}(\bar{u}_n)_i$, $\mathbf{D}(\bar{v}_t)_i$ $(i = 1, 2, 3)$, the values of ε, and the exact value of u_i^e were also given.

P_j	P_1			P_2		
i	1	2	3	1	2	3
			$\varepsilon = 0,1$			
$(\bar{u}_n)_i \cdot 10^3$	—39	88	1391	132	32	1512
$\sqrt{\mathbf{D}(\bar{u}_n)_i} \cdot 10^3$	90	110	120	105	123	132
$(\bar{v}_t)_i \cdot 10^3$	—65	—15	1563	217	2	1795
$\sqrt{\mathbf{D}(\bar{v}_t)_i} \cdot 10^3$	31	32	36	38	35	42
			$\varepsilon = 0,01$			
$(\bar{v}_t)_i \cdot 10^3$	—107	1	1606	110	14	1831
$\sqrt{\mathbf{D}(\bar{v}_t)_i} \cdot 10^3$	20	21	26	13	13	28
$u_i^e \cdot 10^3$	—110	0	1632	113	0	1911

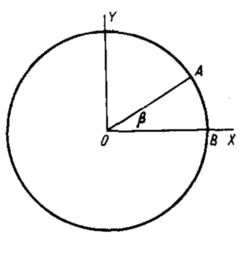

Fig.1

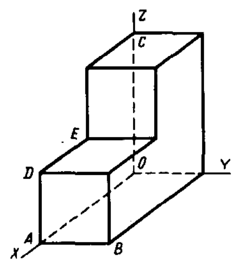

Fig.2

Using these results we can conclude that the estimate (2.2), compared with the estimate (2.1), yields a substantial improvement in convergence.

The algorithm given above was used to find the vector $\mathbf{u}(P)$ in problem (1.1). Let us construct an algorithm for determining the derivatives $u_{i,j} \equiv \partial u_i/\partial x_j$ in the problem (1.1). We can obtain, for the vector

$$\mathbf{\Phi}(P) = (u_{1,1}, u_{1,2}, u_{1,3}\, u_{2,1}, u_{2,2}, u_{2,3}, u_{3,1}, u_{3,2}, u_{3,3})$$

the following integral relation:

$$\mathbf{\Phi}(P) = \frac{1}{4\pi R^2} \int\limits_{S(P,\,R)} B(P,Q)\,\mathbf{u}(Q)\,dS(Q) \tag{2.4}$$

where $B(P,Q)$ is a 9×3-matrix whose elements are found from the solution of the first boundary value problem of the theory of elasticity for a sphere /11/.

Let P denote any point of the region $G \setminus \Gamma_\varepsilon$. The value of $\mathbf{\Phi}(P)$ is found from the realizations of the random quantity

$$\mathbf{\Phi}_\mu(P) = B(P, Q_1)\,A(Q_1, Q_2) \ldots A(Q_{\mu-1}, Q_\mu)\,\mathbf{u}(Q_\mu)$$

where $B(P, Q_1)$ is the matrix appearing in the integral relation (2.4) and $A(Q_{i-1}, Q_i)$ is a matrix of the form (1.3).

3. Use of the integral representations of the solution of the wave equations.

In the theory of elasticity the solution of the wave equation in R^3 can be represented, under the given initial conditions

$$\partial^2\mathbf{u}/\partial t^2 - b^2\nabla^2\mathbf{u} - (a^2 - b^2)\,\mathrm{grad\,div}\,\bar{\mathbf{u}} = \mathbf{K} \tag{3.1}$$
$$\mathbf{u}(x, y, z, 0) = \mathbf{f}(x, y, z),\ \partial\mathbf{u}/\partial t\,|_{t=0} = \boldsymbol{\varphi}(x, y, z)$$

in the form of a sum of integrals up to the quintuple inclusive /12/, and the integrals can be found using the MC method /13/.

We will use the proposed method to solve the test problem (3.1) with the following initial data:

$$\mathbf{K} = \mathbf{A}(a^2 - 1, b^2 - 1)\cos t,\ \mathbf{f} = \mathbf{A}(1, 1),\ \boldsymbol{\varphi} = 0$$
$$\mathbf{A}(\alpha, \beta) = (\alpha \sin x + \beta \cos y,\ \alpha \sin y + \beta \cos z,\ \alpha \sin z + \beta \cos x)$$

In this case $u(x, y, z, t) = A(1, 1)\cos t$ is the solution of the problem. The following estimate is obtained for the solution at the point $x = \pi/2, y = \pi/4, z = \pi/8$ for $N = 50\,000, a = 2, b = 1, t = 1$: $\bar{u} = (0.93; 0.91; 0.22)$ (the exact solution is $u = (0.92; 0.88; 0.21)$).

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ON AN ERROR IN THE THEORY OF THE CONFORMAL MAPPING OF SIMILAR REGIONS AND ITS APPLICATION TO THE FLOW PAST A PROFILE*

A.L. GONOR

The correct value of the peripheral derivative in the conformal mapping of the outsides of similar regions is determined and used in the formula for the velocity distribution over a contour similar to the given profile. The formula contains a correction and examples are given of determining the velocity distribution on an elliptic profile.

When plane fluid flows are investigated, formulas for recomputing the velocity distribution during the passage from the given profile C to a similar profile C_1 (Fig.1) are frequently encountered. The formulas make it possible to alter the hydrodynamic characteristics of a wing. The basic results of this problem are given in /1/, and in all editions of the book /2/.

Let us carry out a critical analysis of the formulas derived, following the account given in /2/. Let the flow pattern past the profile C be known, and the conformal mapping

$$\zeta = F(z, C), \quad F(\infty, C) = \infty \tag{1}$$

be given of the outside of C onto the outside of the unit circle $|\zeta| > 1$, for which, in particular, the correspondence between the points of C and the points of the circumference $\zeta = e^{i\theta}$ ($s = s(\theta)$, s is the arc length along the contour C) is determined.

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